# 2024 r/mathmemes math contest

lets clutch this

April 2024

## 1 Information

1. You will be allotted 48 hours of time to solve the problems and submit your answers via the provided Google Form.
2. Since it would be impractical to restrict, any online calculators or aids will be allowed. However, you should subscribe to the honor code of solving the problems by yourself only.
3. Problems range in (roughly increasing) difficulty from early AMC to mid-late Putnam level.

## 2 Problems

1. $N$ visitors of the LycoReco cafe each give an integer rating from 1 to 10 on Takina Inoue's chocolate parfait, with 1 standing for absolutely atrocious and 10 standing for very delicious. Given that the sum of all ratings given is 83 and the mean of the ratings equals the median of the ratings, what is the sum of all possible values of $N$?
2. Let $k$ be a positive integer, and let $r$ be a random real number selected from Uniform$(0, 1)$. What is the probability that the when $r$ is written in binary, the runs of 0s and 1s never surpass length $k$? (For instance, the number $\frac{1}{5}$ written in binary is 0.0011001100110011..., in which the runs of 0s and 1s never surpass length 2.) Express your answer in terms of $k$.
3. Consider the 4047 logical statements that pertain to persons $P_1, P_2, ..., P_{2024}$:
   - $L_1 = P_1$ has watched Lycoris Recoil
   - $L_2 = P_1$ and $P_2$ have watched Lycoris Recoil
   - $L_3 = P_1, P_2,$ and $P_3$ have watched Lycoris Recoil

   $\vdots \quad \vdots \quad \vdots$
   - $L_{2023} = P_1, P_2, P_3, ...,$ and $P_{2023}$ have watched Lycoris Recoil
   - $L_{2024} = P_{2024}$ has watched Lycoris Recoil
   - $L_{2025} = P_{2023}$ and $P_{2024}$ have watched Lycoris Recoil
   - $L_{2026} = P_{2022}, P_{2023},$ and $P_{2024}$ have watched Lycoris Recoil

   $\vdots \quad \vdots \quad \vdots$
   - $L_{4046} = P_2, P_3, P_4, ..., P_{2022}, P_{2023},$ and $P_{2024}$ have watched Lycoris Recoil
   - $L_{4047} = P_1, P_2, P_3, ..., P_{2022}, P_{2023},$ and $P_{2024}$ have watched Lycoris Recoil

   Suppose exactly $k$ of the statements in the set $\{L_1, L_2, L_3, ..., L_{4047}\}$ are true (and the rest are false). How many distinct possible values of $k$ are there?

4. 12 people, including Alice and Bob, are standing in a circle. Alice leaves the circle first, and each subsequent second, one of the two people who were in the previous second adjacent to the person who just left (chosen at random) leaves, until there is one last person remaining, in which they also leave. If Alice and Bob were originally standing diametrically opposite to each other, what is the probability that Bob is the last to leave the circle?

5. 2023 light switches corresponding to 2023 respective lightbulbs lie in a row, all initially turned off. Toggling each light switch will cause no change in the state of the lightbulb it corresponds to but will toggle (flip) the states of all 2022 other lightbulbs. How many possible binary configurations of the 2023 lightbulbs are attainable through some finite sequence of toggling light switches one at a time?

6. Let $f(n) = \int_0^1 x^n e^x dx$. Compute $\lim_{n\to\infty} \left| \frac{f(n)-n!}{n!e} \right|$.

7. Mark writes a sequences of integers on the board as follows: He first writes the number 1 (this is the first iteration). Each subsequent iteration, he first appends the number that is one more than the maximum number on the board so far to the list and then appends the list of numbers that was on the board at the end the previous iteration to the list. For example, after three iterations, Mark will have written the numbers $1, 2, 1, 3, 1, 2, 1$, in that order. Let $f(n)$ be the $n$th number Mark writes on the board, $f^*(n)$ denote $\lim_{k\to\infty} \underbrace{f(f(f...(f(n)...))}_{\text{k compositions}}$ and let $N$ be the number of integers in the range $1 \leq n \leq 2^{2^{31}} - 1$ such that $f^*(n) > 1$. If $B(n)$ denotes the number of ones in the binary representation of $n$, what is $B(B(N))$?

8. A dormitory hall has 10 students, each with their own rooms. There's a rule that two students can only switch rooms if they are friends. Let $M$ be the least possible number of pairs of students that are friends such that every possible assignment of the 10 students to the 10 rooms could possibly be attained after a finite number of switches. (For example, if A and B are friends, B and C are friends, but A and C are not friends, then there are two pairs of friends among A, B, and C.) How many distinct friendship configurations of the 10 students with exactly $M$ pairs of students that are friends satisfy the condition that every possible rooming assignment is attainable after a finite number of switches?

9. 2024 Lycoris Recoil fans $L_1, L_2, L_3, ..., L_{2024}$ are at the Lycoris Recoil event at an anime convention, seated around a circular table. Each fan $L_j$ is asked to state how much they like Chisato on a scale of 0 to 10 and then record their opinion (which can be any real number $C_j$ from 0 to 10 inclusive) on a piece of paper. Each $L_j$ then erases their original number $C_j$ and replaces it with $T_j = \log_2 \frac{C_j}{5}$ if $0 \leq C_j \leq 5$ and $T_j = -\log_2 \left( \frac{10-C_j}{5} \right)$ if $5 < C_j \leq 10$. The organizers of the event then calculates and records the sum of the $T_j$ for each set of $k$ adjacently seated people around the circular table. After the original pieces of paper are discarded, for how many distinct values of $1 \leq k \leq 2024$ is it still always possible to uniquely deduce exactly how much each Lycoris Recoil fan liked Chisato from the organizers' recorded sums?

10. Let $p$ be a prime, $n$ be a positive integer, and denote $S_{n,p}$ as the set of all possible remainders that can be obtained when dividing a perfect square by $p^n$. What is $\lim_{p\to\infty} \lim_{n\to\infty} \frac{|S_{n,p}|}{p^n}$?

11. Define $f(a) = \sum_{n=0}^{\frac{a}{3}} (-1)^n 2^{a-3n} \binom{a-2n}{n}$. What is $\lim_{a\to\infty} \frac{f(a+1)}{f(a)}$?

12. There are an infinite number of universes, all which initially involve Chisato and Takina standing exactly 2 kilometers apart on a line. In universe $k$, Chisato first travels $A(k, 1)$ kilometers towards Takina. Then, Takina travels $A(k, 2)$ kilometers towards Chisato. Then, Chisato travels $A(k, 3)$ kilometers towards Takina. In general, in universe $k$, the two alternate traveling $A(k, n)$ kilometers towards each other, until they meet each other, when they then will confess their love to each other, where $A(k, n)$ is recursively defined as $A(1, n) = \frac{n}{2^n}$ and $A(k, n) = \sum_{i=1}^{n} A(k-1, i) \cdot 2^{i-n-1}$ for all $k \geq 2$.

    Let $R(k)$ denote the ratio to how far Chisato has traveled to how far Takina has traveled by the time they confess to each other in universe $k$. What is the least value of $k$ such that

$$R(k) \leq 1 + \frac{1}{10^{10}}?$$